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Research Article

Geometric Transitions Applied to Classification of Invasive Developmental Disorders

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Abstract

This manuscript has two purposes. The first objective is to fill a mathematical gap in the coupling formulation of two discrete random walk sets by constructing the matrix that quantifies the coupling for walking pairs. The second is to report the observation of two phenomena: withdrawal and approach. As the γ coupling intensity grows, we observe the formation of objects with distinct fractal dimensions, which characterize geometric transitions in random walks as a function of the coupling probability. We observed two geometric transitions. One transition occurs when the system transitions from clearance behavior to approach behavior, another occurs when random walkers are in the approach region. So, we make an analogy between random walks and pervasive developmental disorders, classifying a set of random walks as healthy and others with pervasive developmental disorder (PDD). We note that random walks have critical mutual influence, when $\gamma = 0.625$ the coupling is maximum, where we notice the largest reduction in observations of the behavior of the individual with PDD. besides, for the set of random walks that represent the healthy set, even without PDD, their actions are influenced by individuals with PDD which may lead to a misdiagnosis.

Keywords: random walks; coupling; diffusion

Abbreviations: PPD: Pervasive developmental disorder. ERW: Elephant Random Model.

Introduction

In 1905, the random walk theory was born from K. Pearson's publication in the journal Nature. From this starting point, random walk theory has become a widely used tool for tackling problems in physics (Diniz et al., 2017), Economics (Masoliver et al., 2003), Engineering (McCarthy, 1993) (Klages, 2008), Biology and Medicine (Moura et al., 2018) (Cressoni et al., 2012).

In 2004, G.M. Schütz and S. Trimper discovered a new class of random walks. The class of random walks with memory. This random walk class has the feature of retrieving every step taken in the past with equal probability. This feature earned him the Elephant Random Walk model (ERW). Some other models were built from the ERW model, such as the Alzheimer random walk model (Cressoni et al., 2007), the Gaussian memory profile random walk model (Borges et al., 2012), the exponential memory profile random walk model (Alves et al., 2014), the exponential memory profile random walk model (Moura et al., 2016) and the random walk model with binomial memory profile (Diniz et al., 2017).

In this class of random walks memory is an important feature. The memory is formed by a set of random variables: $\sigma_{t'}$, in which t' is the uniformly chosen time in the ERW model. Every instant of time t, the elephant's decision depends on its entire history retrieved from a uniform distribution. The probability of recovering a past action is $1/t$ with t as the current time. At each moment the decision is recorded in memory, such property attributes a non-Markovian characteristic to the random walk. The walker walks one step to the right or one to the left, as in a one-dimensional Markovian random walk. The equation of stochastic evolution is

$$X_{t+1} = X_t + \sigma_{t+1} \quad (1)$$

For a while $t + 1$, the variable σ_{t+1} takes the value $+1$ (-1) when the walker performs a step to the right (left). Remembering that memory consists of a set of random variables $\sigma_{t'}$ for the period of time $t' < t$. The dynamics of this process occurs as follows:

a) In the period of time $t + 1$ a number t' of the set $1,2, \dots, t$ is randomly chosen with uniform probability 1/ t

b) σ_{t+1} is determined stochastically by, $\sigma_{t+1} = \sigma_{t'}$ with probability p e $\sigma_{t+1} = -\sigma_{t'}$ with probability $1 - p$.

The first step is performed according to a specific rule. In the instant $t = 1$; the walker is in positionX_0, moves to the right$\sigma_1 = +1$ with probability of q to the left $\sigma_1 = -1$ with probability $1 - q$. So it follows that the stochastic evolution equation is:

$$X_t = X_0 + \sum_{t'=1} \sigma_{t'} \quad (2)$$

The parameter is the probability of the walker repeating a past action at the instant of time t'. For $(p > 1/2)$, the walker exhibits persistent behavior; for $(p < 1/2)$ the walker exhibits anti-persistent behavior; for $(p = 1/2)$, the random walk is Markovian. There are two extreme points, $(p = 0)$ and $(p = 1)$, for which random walks exhibit maximum behavior. At the point, the most anti-persistent behavior occurs; at the point: $p = 1$,the most persistent behavior occurs: hiker moves deterministically. The class of random walks with memory exhibit strong dependence on the first step. The first step is macroscopically relevant and may influence the diffusion regimes measured by the Hurst exponent (Hurst et al., 1969).

The first moment of the position is:

$$\langle x(t)\rangle = \frac{\delta}{\Gamma(\lambda+1)} t^{\lambda} \tag{3}$$

In which $\delta = 2q - 1$, $\lambda = 2p - 1$ and Γ is the gamma function. The parameters δ and λ are set in the range $[-1,1]$.

The second moment of the position is given by:

$$\langle x^2(t)\rangle = \begin{cases} \frac{t}{3-4p} & ,p < \frac{3}{4} \\ tlnt & ,p = \frac{3}{4} \\ \frac{t^{4p-2}}{(3-4p)\Gamma(4p-2)} & ,p > \frac{3}{4} \end{cases} \tag{4}$$

Observe that $(p < 3/4)$ the second moment, equation 4 depends linearly on and the diffusion is ordinary, for $(p > 3/4)$ diffusion is characterized as superdiffusive. In the point $(p = 3/4)$ the second moment is described by a logarithmic function of time (Schutz e Trimper, 2004).

Methodology

The ERW Two Dimensional Model

Two random ERW walkers, 1 and 2, walk along two distinct, perpendicular coordinate axes, just like: x_1 and x_2, respectively. In the instant of time t, the position of the first walker is denoted by X_t^1. He recovers steps from his own memory as well as steps from the history of the second PAE, labeled X_t^2. The position of the i-nth random walker is quantified by the stochastic equation

$$X_{t+1}^i = X_t^i + \sigma_{t+1}^i \tag{5}$$

with $i = 1,2$. Microscopic dynamics follows the following rules:

1) in the period of time $t + 1$, the elephant: i choose one of the indexed elephants $k = 1,2$ with probability γ^i_k. This probability must to satisfied the following relationship $\gamma^i_1 + \gamma^i_2 = 1$;
2) in the period of time: $t + 1$, a period of time: t' is uniformly chosen from the set $\{1,2,3, \ldots, t\}$;
3) in the period of time: $t + 1$, the step of i-th of the elephant is $\sigma_{t+1}^i = +\sigma_{t'}^k$, with probability p_k^i e $\sigma_{t+1}^i = -\sigma_{t'}^k$, with probability $1 - p_k$. So we note that the probability is:
$$P[\sigma_{t+1}^i = \pm\sigma_{t^F}^k \vee \sigma_{t^F}^k] = \frac{1}{2}[1 + (2p_k^i - 1)\sigma_{t+1}\sigma_{t'}^k] \tag{6}$$
4) The probability of the first step is:
$$P[\sigma_1^i = \pm 1 \vee sentidodek] = \frac{1}{2}[1 + (2q_k^i - 1)\sigma_1^i] \tag{7}$$

The probability of the step of the i-nth that is walking in time $t + 1$, $\sigma_1^i = \sigma$, that comes from the spectrum of possibilities of the set $\{\sigma_{t'}^1, \sigma_{t'}^2\}$, is

$$P[\sigma_{t+1}^i = \sigma \vee \sigma_{t^F}^{1,2}] = \frac{1}{2t}\sum_{k=1}^{2}[1 + (2p_k^i - 1)\sigma\sigma_{t^F}^k]\gamma_k^i \tag{8}$$

In which the parameter γ_k^i is probability of coupling of the i-nth elephant in relation to
k-nth elephant. So the probability of the first step is:

$$P[\sigma_1^i = \pm 1] = \frac{1}{2}\sum_{k=1}^{2}[1 + (2q_k^i - 1)\sigma]\gamma_{t^F}^k \tag{9}$$

From equation 7 it is possible to calculate the probability of $\sigma_1^i = \pm 1$, i.e., $P[\sigma_{t+1}^i = \sigma \vee \sigma_1^1, \ldots, \sigma_t^1; \sigma_1^2, \ldots, \sigma_t^2] = P[\sigma_{t+1}^i = \sigma \vee \{\sigma_{1,2,\ldots,t}^{1,2}\}]$

which is:

$$P[\sigma_{t+1}^i = \sigma \vee \{\sigma_{1,2,\ldots,t}^{1,2}\}] = \frac{1}{2t} + \frac{\sigma}{2}\sum_{k=1}^{2}\alpha_t^i\gamma_k^i x_t^k \tag{10}$$

in which $\alpha_t^i = 2p_k^i - 1$ e $x_t^k = X_t^k - X_0^k$.

The conditional displacement of the i-nth elephant is:

$$\langle\sigma_{t+1}^i = \sigma|\{\sigma_{1,2,\ldots,t}^{1,2}\}\rangle = \sum_{k=1}^{2}\frac{\alpha_k^i\gamma_k^i x_t^k}{t} \tag{11}$$

So from equation 11 we get the recursive relation of the first moment of position:

$$\langle x_{t+1}^i\rangle = \sum_{k=1}^{2}\left(\delta_{ki} + \frac{\alpha^k\gamma^k}{t}\right)\langle x_t^k\rangle \tag{12}$$

Another offset parameter, in this case for the first step, needs to be set as: $\beta_k^i = 2q_k^i - 1$. Through this definition we can find the initial offset for the walking of index: i

$$\langle x_1^i\rangle = \sum_{k=1}^{2}\beta_k^i\gamma_k^i \tag{13}$$

The memory-coupled elephant random walk model was published in (Marquioni, 2019). Based on it, we reproduce the above results as a basis for building our model. Our model will use the memory coupling mechanisms introduced numerically and analytically in (Moura et al., 2018) and (Marquioni, 2019), in this exact order.

The Random Walk Model with PDD Characteristic

The PDD analogy for constructing the random walk model was first presented in (Moura et al, 2018). Others work emerged as a consequence, investigating new phenomenologies and introducing new mathematical tools (Moura-Ramos-Ramos, 2018), (Moura, 2019), (Marquioni, 2019).

In (Moura et al, 2018), PDD-type random walks were first introduced. In this model two random walks are unilaterally coupled, that is, one set of random walkers is influenced by the microscopic decisions of the other set of walkers, which are independent.

The first set of random walks is defined as independent, defined as the ERW model, which has the label of “Professor”. This model makes decisions from its own history. The second set of randoms, labeled “Student”, because it has the characteristic of maximum persistence in its microscopic decisions was defined as autistic, which can be influenced by its microscopic decisions according to its own memory or according to the memory of the ERW model. Decisions are retrieved from your memory (Professor's memory) with probability (γ) $((1-\gamma))$. A superdiffusive regime was observed for all feedback parameter values in the range: p Є [0,1] with $(\gamma \neq 0)$. This phenomenon has never been observed in another model derived from the ERW randoms. For $(\gamma = 0)$ walkers diffuse independently.

In (Moura-Ramos-Ramos, 2018), it was proposed to answer the following questions: How can the Professor influence and be influenced by the Student's microscopic decisions? If both learn from each other by making microscopic decisions that influence each other, what can we call a Professor process and Student process? To solve this apparent teaching-learning paradox, by construction, a bilateral memory coupling was performed. The Professor can learn (not learn) from the Student's decisions. Similarly, the Student can learn (not learn) from the Professor's decisions.

From the next section, we will use the mathematical formulation presented in (Marquioni, 2019), (Bercu, 2018) and (Bercu et al, 2019) to construct the coupling matrix that describes mutual influence of random walks. This is one of the main objective of this work.

The coupling

At this point, we will label "Professor" as set of random walks 1 and "Student" as set of random walks 2. Random walks 1 and 2 walk on different coordinate axes, the randoms with number 1 walk on axis x_1, perpendicular to the axis x_2 where the 2 randoms 2 walk. Each walker's microscopic steps may influence each other's present state. The probability of walker decisions: i to be influenced by walker decisions: j is quantified by the coupling coefficient γ^i_j.

Coupling coefficients are listed as follows:

$$\begin{cases} \gamma^1_1 = 1-\gamma^1_2 \\ \gamma^1_2, \forall 0 \le \gamma^1_2 \le 1 \\ \gamma^2_1, \forall 0 \le \gamma^2_1 \le 1 \\ \gamma^2_2 = 1-\gamma^2_1 \end{cases} \quad (15)$$

According to equation 15, the first moment of position is given by the relation below:

$$\begin{pmatrix} \langle x^1_{t+1} \rangle \\ \langle x^2_{t+1} \rangle \end{pmatrix} = \frac{1}{t} \begin{pmatrix} t+(1-\gamma^1_2)\alpha^1_1 & \gamma^1_2\alpha^1_2 \\ \gamma^2_1\alpha^2_1 & t+(1-\gamma^2_1)\alpha^2_2 \end{pmatrix} \begin{pmatrix} \langle x^1_t \rangle \\ \langle x^2_t \rangle \end{pmatrix} \quad (16)$$

According to equation 16, under the condition that there is a double coupling of 2 to 1, as well as 1 to 2. There are two cases to consider for coupling: the symmetrical case ($\gamma^1_2 = \gamma^2_1$) and the non-symmetrical case ($\gamma^1_2 \neq \gamma^2_1$). In the symmetrical case walkers 1 and 2 have the same learning probability ($\gamma^1_2 = \gamma^2_1$). In the non-symmetrical case 1 or 2 may have probability functions that grow faster than the other. For example: ($\gamma^1_2 > \gamma^2_1$)when walker 1 is more likely to modify his microscopic decisions according to walker 2's actions, and ($\gamma^1_2 < \gamma^2_1$) otherwise. Let's address the case where($\gamma^1_2 = \gamma^2_1 = \gamma$) with $0 \le \gamma \le 1$. Besides, let's vary a single feedback probability parameter by calling $\alpha^1_1 = \alpha^1_2 = \alpha^2_1 = \alpha^2_2 = \alpha$ com $-1 \le \alpha \le 1$, in which $\alpha = 2p-1$.

Following these changes we rewrite the matrix equation of the first moment of position as:

$$\begin{pmatrix} \langle x^1_{t+1} \rangle \\ \langle x^2_{t+1} \rangle \end{pmatrix} = \frac{1}{t} \begin{pmatrix} t+(1-\gamma)\alpha & \gamma\alpha \\ \gamma\alpha & t+(1-\gamma)\alpha \end{pmatrix} \begin{pmatrix} \langle x^1_t \rangle \\ \langle x^2_t \rangle \end{pmatrix} \quad (17)$$

We performed coupling between the aforementioned random walkers. The Professor with label 1 and the Student with label 2. We performed measurements of the first moment of Hurst position and exponent for various coupling intensities values: γ.

Discussion & Conclusion

We measured the effects of random walkers' coupling from fractal dimension measurements. To perform the measurements of the physical observables of the diffusive processes we proceeded with the ordinary analysis of the first moment of position $\langle x \rangle$ and measurements of the fractal dimension (D). In problems that can be modeled by random walks the diffusive regimes can be measured by the exponent H. The diffusive behavior can be quantified from the asymptotic scale law of the mean square deviation of position in relation to time that is $\langle x^i_t \rangle^2 = A_i t^{2H_i}$ with i=1,2 related to walkers labeled 1 and 2. Where A_i is a constant of proportionality and H_i is the Hurst exponent of the i-nth walker. Diffusive regimes can be classified into sub-diffusive ($H < 1/2$), ordinary diffusive ($H = 1/2$) and super diffusive ($H > 1/2$) (Hurst et al., 1969). The fractal dimension is related to the Hurst exponent according to the following equation$D_i = \delta + 1 - H_i$, in which δ is dimension in Euclidean space and D_i is the fractal dimension. The problem is modeled from a two-dimensional random walk ($\delta = 2$).

In Figure 1, measurements of the difference in the first moment position of random walkers are displayed. $\langle x^1_{t+1} \rangle - \langle x^2_{t+1} \rangle$. The typical behavior of the first position moment for the random walkers 1 and 2 coupling problem, as a function of feedback and coupling: p and γ, respectively; It is characterized by two phenomena. The phenomena of distance and approach. Measurements were made for the feedback parameter $0 \le p \le 1$coupling parameter varying in range: $0 \le \gamma \le 1$. Our observations show that the clearance prevails over the approximation phenomenon to typical values of: $\gamma < 0.2$. In $\gamma = 0$, the distance appears with greater intensity, presenting average differences of the order of steps. As the coupling grows: $\gamma > 0$, The distance between walkers is reduced. In $\gamma \approx 0.2$, this average difference is reduced to the order of 10^0 steps. So from this point $\gamma > 0.2$, the approximation phenomenon prevails over the removal phenomenon, becoming stronger as $\gamma \rightarrow 1$ memory coupling becomes more intense.

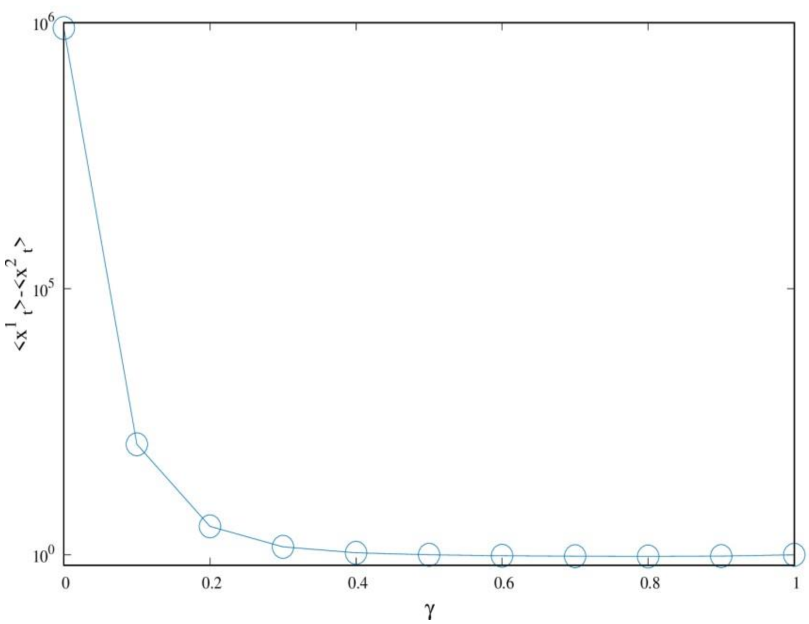

Figure 1: Typical measurements of the average distance difference between walkers 1 and 2 $\langle x^1_{t+1} \rangle - \langle x^2_{t+1} \rangle$ for $p = 1/2$. Measurements were made for coupling parameter varying in the range: $0 \le \gamma \le 1$.

In Figure 2, fractal dimension measurements are displayed by setting the feedback parameters to a single value: $p^1 = p^2 = p$. The curves were obtained for the quantitative values of the feedback parameter equal to $p = 0.7$ and $p = 0.9$ and for the coupling parameter in the range of $0 \leq \gamma \leq 1$. The fractal dimension curves of the random walkers were labeled as follows: the label RW1 for walker 1 and RW2 for the second walker. The curves at the top of figure 2 are for $p = 0.7$. We note that the rapid convergence of the fractal dimensions occurs ($D^1 \approx D^2$) even when the system is subjected to a slight increase in the coupling parameter : γ. Insofar as: γ gets stronger the coupling becomes, and the system no longer exhibits a characteristic behavior of the withdrawal phenomenon, transitioning to a behavior that characterizes the approach phenomenon. Approximation is a phenomenon characterized by fractal dimension measurements that converge to a single value for both random walkers. $D^1 = D^2$. In $\gamma = 0$, random walkers with label 1 exhibit fractal dimension: $D^1 \approx 2.5$, characteristic of a dimensioned object between a two-dimensional object and a three-dimensional object, whereas the .random walkers with label 2 have $D^2 = 2$ of a two-dimensional object. As the coupling increases: $\gamma > 0$, the measures of D^1 decrease and the D^2 measures increase. After a variation in the coupling parameter, we note that the fractal dimensions of two random walkers converge to typical congruent values: $D^1 = D^2$. At the bottom of Figure 2, for $p = 0.9$ we noted the same convergence behavior for fractal dimension measurements. We observed a behavioral change in the fractal dimension measurements of walker 1 in the $\gamma < 0.2$. The D^1 measures show that there is an oscillation in the region of $\gamma < 0.2$, which is characterized by the increase of D^1, for a small increase $\gamma > 0$, of a two-dimensional object ($D^1 = 2$) for an object with dimension between two-dimensional and three-dimensional, with fractal dimension ($D^1 \approx 2.3$). Increasing the values of γ, we noted the quantitative reduction of D^1. And also increasing γ, in the region $\gamma > 0.2$, approximately we noted the convergence of fractal dimension measurements $D^1 = D^2 = D$. After the convergence in fractal dimension measurements, which we noted only by D, the dimension of random walks undergo a transition from a object with dimension $D > 2$ for a two-dimensional object $D = 2$.

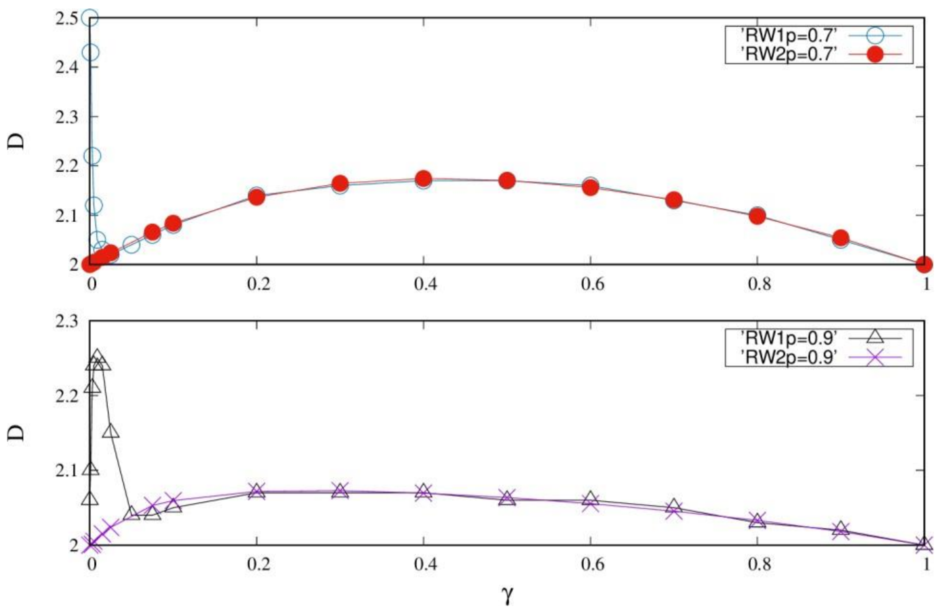

Figure 2: Fractal dimension measurements (D) walkers 1 and 2. Measurements were made for the feedback parameter equal to $p = 0.7$ and $p = 0.9$ with coupling parameter varying in the range of $0 \leq \gamma \leq 1$. We use the label RW1 for walker 1 and RW2 for the second walker.

Conclusion

We proceed with the mathematical formulation of the coupling of two discrete random walk sets with two-dimensional coupling. We built the matrix that quantifies the coupling for discrete random walking pairs. From the coupling matrix, we perform numerical experiments to obtain the physical observables.

We perform numerical simulations of memory-coupled walks to answer the following question: How can the Professor influence and be influenced by the Student's microscopic decisions and vice versa? If both walkers learn from each other, how can we support the analogy of teaching and learning and call one process Professor and Student process? This apparent teaching-learning paradox was resolved by using the bilateral memory coupling technique (Moura-Ramos-Ramos, 2018).

We observed two physical phenomena related to coupling: the phenomenon of distance and approach. We noted that in the distance phenomenon, random walkers diffuse with less dependence, while greater variations in the coupling parameter are accompanied by greater dependence among random walkers. This dependence was quantified from the measurements of the first moment of position and fractal dimension of walkers 1 and 2. With a slight increase in the variation of the coupling parameter the system converges to the measurements of the first moment of position and to the fractal dimension measurements. More severe mental disorders are related to smaller fractal dimensions, while larger fractal dimensions can be used to classify less severe degrees of invasive developmental disorders. The distance and approach phenomena show rapid convergence with variations in the coupling probability. These phenomena carry the information that even a walkable individual, walker 1, in giving in and receiving information from an individual with some type of invasive developmental disorder, walker 2, can be influenced by its actions. The bond for mutual learning is quantified by the probability of coupling, which does not classify the reasons for justifying the intensity of coupling, we take as random because we do not know them, we can quantify them from the coupling parameter. We observed that walker 2, too, is influenced by walker 1's actions. Coupling has as a consequence the quantitative increase of fractal dimension for walker 1, which is associated with lower observations of the behavior of an individual with some invasive developmental disorder.

Conflict of Interest

There are no conflict of interests.

Acknowledgements

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